TOPOLOGY - III, SOLUTION SHEET 9

Exercise 1. First we look at the composition $\pi_n(X) \xrightarrow{h} H_n(X) \xrightarrow{\phi_*} H_n(Y)$. Let $[f] \in \pi_n(X)$. Then from the definition of the Hurewicz map it follows that $h([f]) \in H_n(X)$ is given by $f_*(u)$ where $f_* : H_n(S^1) \to H_n(X)$ is the induced map on homology. Then $\phi_*(h([f])) = (\phi_* \circ f_*)(u)$. Now we compute the composition $\pi_n(X) \xrightarrow{\phi_*} \pi_n(Y) \xrightarrow{h} H_n(Y)$. We have that $\phi_*([f]) = [f \circ \phi]$ and hence $h(\phi_*([f])) = (f \circ \phi)_*(u) = (f_* \circ \phi_*)(u)$. This shows that the diagram in question commutes. $\square$

Exercise 2. (1) If U_2 is contractible then $\pi_1(U_2)$ is trivial and hence it follows from the pushout diagram in Van Kampen's theorem that $\pi_1(X)$ has the same universal property as that of $\pi_1(U_1)/\langle i_1(\pi_1(U_1 \cap U_2) \rangle$. $\square$

(2) We compute the fundamental group of $(T^2)^{\#n}$ using it's planar diagram given by $\Sigma = a_1 b_1 a_1^{-1} b_1^{-1} ... a_n b_n a_n^{-1} b_n^{-1}$. One can also refer to the applications of Proposition 1.26 in Hatcher's book for a similar proof. The case for $(\mathbb{RP}^2)^{\#n}$ goes through in the same way and so we leave it to the reader.

As hinted, let $V_1, V_2 \subset \Sigma$ opens, where V_1 is given by Σ punctured at the centre and V_2 is a small open disk around the centre. Let $q : \Sigma \to (T^2)^{\#n}$ be the quotient map. Note that q is a homeomorphism on $\Sigma - \partial\Sigma$ and that $q(\partial\Sigma)$ is the wedge-sum of $2n$ copies of S_1. Define open neighbourhoods of Σ given by $U_1 := q(V_1)$ and $U_2 := q(V_2)$. Further note that $q(V_2) \cong U_2$ is contractible and that $U_1 = q(V_1)$ deformation retracts onto $q(\partial\Sigma)$ since V_1 deformation retracts onto $\partial\Sigma$. Moreover $U_1 \cap U_2 \cong V_1 \cap V_2$ is a punctured open disk. By (1), we have that $\pi_1((T^2)^{\#n}) = \pi_1(U_1)/\langle i(\pi_1(U_1 \cap U_2)) \rangle$, where i is the inclusion of $U_1 \cap U_2$ in U_1. So we have that $\pi_1(U_1)$ is the free group on $2n$ letter $a_1, b_1, ..., a_n, b_n$ since it the the wedge sum of $2n$ circles. Also the fundamental group of $U_1 \cap U_2$ is the free group on one generator since $U_1 \cap U_2$ is a punctured disk. One notes that this generator gives the same class as $a_1 b_1 a_1^{-1} b_1^{-1} ... a_n b_n a_n^{-1} b_n^{-1}$ in $\pi_1((T^2)^{\#n})$. (up to an isomorphism of fundamental groups given by a change of base points). Hence we obtain the desired isomorphism

$$\pi_1((T^2)^{\#n}) = \langle\, a_1, b_1, ..., a_n, b_n \mid [a_1, b_1] \cdot \ldots \cdot [a_n, b_n] = 1 \,\rangle.$$

(3) We use the fact that the abelianisation of a group presented as $\langle s_1, s_2, ..., s_n \mid r_1, .., r_k \rangle$ is isomorphic to

$$\frac{\mathbb{Z} \cdot s_1 \oplus \mathbb{Z} \cdot s_2 \oplus ... \oplus \mathbb{Z} \cdot s_n}{\langle r_1, ..., r_k \rangle}.$$

That is the free abelian group on the generators quotiented by the same relations.

(a) Since the commutators $[a_i, b_i]$ are equal to 1 in the abelianisation, there are no non-trivial relations and hence $H_1((T^2)^{\#n}) = \mathbb{Z}^{2n}$.
(b) We have that $H_1((\mathbb{RP}^2)^{\#n}) = \mathbb{Z}^n / \langle (2, 2, ..., 2) \rangle \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}/2\mathbb{Z}$.